PERMUTATION PRESENTATIONS OF MODULES OVER FINITE GROUPS.

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ABSTRACT. We introduce a notion of permutation presentations of modules over finite groups, and completely determine finite groups over which every module has a permutation presentation. To get this result, we prove that every coflasque module over a cyclic p-group is permutation projective.

1. INTRODUCTION

In this paper, we introduce a notion of permutation presentations of modules over finite groups, and completely determine finite groups over which every module has a permutation presentation. This result is used in [K] to solve a problem on C^*-algebras. Throughout this paper, Γ denotes a finite group. Every modules are left modules.

Definition 1.1. A $\mathbb{Z}\Gamma$-module F is called a *permutation module* if we can find a basis of F as a $\mathbb{Z}$-module which is permuted by the action of Γ.

For a Γ-set X, the free abelian group $\mathbb{Z}[X]$ whose basis is given by $\{[x]\}_{x \in X}$ is a permutation $\mathbb{Z}\Gamma$-module, and every permutation module is isomorphic to this type of module.

Definition 1.2. A *permutation presentation* of a $\mathbb{Z}\Gamma$-module M is an exact sequence

$$0 \longrightarrow F \longrightarrow F \longrightarrow M \longrightarrow 0$$

of $\mathbb{Z}\Gamma$-modules where F is a permutation $\mathbb{Z}\Gamma$-module. A permutation presentation is said to be *countable* when F in the sequence above is countable.

If a $\mathbb{Z}\Gamma$-module M has a countable permutation presentation, then M is countable. The following proposition, which will be proved in Section 3, says that the converse also holds.

Proposition 1.3. *If a countable $\mathbb{Z}\Gamma$-module has a permutation presentation, then it has a countable permutation presentation.*

For a Γ-set X, an endomorphism φ of the permutation $\mathbb{Z}\Gamma$-module $\mathbb{Z}[X]$ defines a $X \times X$-matrix $A = (A_{x,x'})_{x,x' \in X}$ with $A_{x,x'} \in \mathbb{Z}$ by $\varphi([x]) = \sum_{x' \in X} A_{x,x'}[x']$. This matrix A satisfies that $\{x' \in X \mid A_{x,x'} \neq 0\}$ is finite for all x, and $A_{x,x'} = A_{\gamma(x),\gamma(x')}$ for all x, x' and $\gamma \in \Gamma$. Conversely such a Γ-invariant $X \times X$-matrix A with integer entries defines an endomorphism of $\mathbb{Z}[X]$. Therefore a permutation presentation of a $\mathbb{Z}\Gamma$-module M is

presented by a Γ-set X and a Γ-invariant $X \times X$-matrix A with integer entries such that $\ker A = 0$ and $\operatorname{coker} A \cong M$ as $\mathbb{Z}\Gamma$-modules. We will use this observation in [K].

The following is the main theorem of this paper.

Theorem 1.4. *For a finite group Γ, every $\mathbb{Z}\Gamma$-module has a permutation presentation if and only if every Sylow subgroup of Γ is cyclic.*

There exists an explicit description of finite groups all of whose Sylow subgroups are cyclic. Such a group is isomorphic to a semi-direct product $\Gamma = (\mathbb{Z}/m\mathbb{Z}) \rtimes (\mathbb{Z}/n\mathbb{Z})$ such that m and n are relatively prime integers and that $\mathbb{Z}/m\mathbb{Z}$ is the commutator group of Γ (for detail, see [Ro, 10.1.10] for example). By Theorem 1.4, we see that for a finite cyclic group Γ every $\mathbb{Z}\Gamma$-module has a permutation presentation. We remark that when the order of Γ is prime, this fact follows from [Sp, Theorem 1.3].

The condition that a given $\mathbb{Z}\Gamma$-module M has a permutation presentation is much weaker than it looks. This is equivalent to the condition that the permutation projective dimension of M is 0 or 1, which was introduced by Arnold in [A1] (see the beginning of Section 3 in this paper). In [A2], a finite group Γ was said to have a global permutation projective dimension 1 if every *finitely generated* $\mathbb{Z}\Gamma$-module M has a permutation projective dimension 0 or 1. Using the results in [EM], Arnold showed that the condition that every Sylow subgroup is cyclic characterizes finite groups with global permutation projective dimension 1 in his sense (see the remark after [A2, Definition 1.5]). On the other hand, one can define a global permutation projective dimension of a group Γ using all $\mathbb{Z}\Gamma$-modules, and Theorem 1.4 with the remark above shows that the condition for a finite group to have a global permutation projective dimension 1 in this sense is the same as the one considered by Arnold. Thus two notions of global permutation projective dimension 1 coincide. The author does not know whether this coincidence happens for higher dimensions (see [A1] for the precise definition of permutation projective dimension).

In order to prove Theorem 1.4, we can use many arguments in [A1] and [EM] without big changes. However, in the one step in [EM], Endo and Miyata seemed to use a property of modules over the Dedekind domain $\mathbb{Z}[\zeta_q]$ which is valid only for *finitely generated* modules, and hence we cannot use their argument. We need a new idea to complete this step in our case. To express this step, we introduce the following notions (see [L, Subsection 2.10]).

Definition 1.5. A $\mathbb{Z}\Gamma$-module M is said to be *permutation projective* if it is isomorphic to a direct summand of some permutation module, and *coflasque* if M is free as a $\mathbb{Z}$-module and $H^1(\Gamma', M) = 0$ for all subgroups Γ' of Γ.

For a definition and results of the group cohomologies $H^i(-, -)$, we consult the books of Brown [B] and Serre [Se]. By Shapiro's lemma, every permutation projective $\mathbb{Z}\Gamma$-module is coflasque (see Remark 2.3). Now the following theorem is the most difficult part in our proof of Theorem 1.4.

Theorem 1.6. *Let Γ be a cyclic p-group. Then every coflasque $\mathbb{Z}\Gamma$-module is permutation projective.*

This theorem seems to have its own importance. Recently, several authors investigate the structures of $\mathbb{Z}\Gamma$-modules which is free as a $\mathbb{Z}$-module. Such a module is called a *generalized lattice* in [BCK]. The proof of Theorem 1.6 is inspired from [BCK], and is more elementary (though longer) than the arguments in [EM].

This paper is organized as follows. In Section 2, we give a cohomological obstruction for a module to have permutation presentations. We also prove the "only if" part of

Theorem 1.4. In Section 3, we sketch the proofs of propositions which were proved in [A1] and [EM] for finitely generated modules explicitly or implicitly. These propositions give the induction of Theorem 1.4 to Theorem 1.6. We also prove Proposition 1.3. Section 4 is a preparatory section for the proof of Theorem 1.6, and in Section 5 we prove Theorem 1.6 and thus complete the proof of Theorem 1.4.

Acknowledgments. In March 2006, the author held a seminar about this paper at University of Tokyo, and he is grateful to the participants for asking stimulating questions and simplifying some arguments in an earlier draft of this paper.

2. An obstruction to have permutation presentations

In this section, we give a cohomological obstruction for a $\mathbb{Z}\Gamma$-module to have permutation presentations, and prove the "only if" part of Theorem 1.4.

Recall that the exponent $\exp(\Gamma)$ of a finite group Γ is the smallest positive integer n satisfying $\gamma^n = 1$ for all $\gamma \in \Gamma$. The following lemma is easy to see, hence we omit the proof.

Lemma 2.1. *The exponent* $\exp(\Gamma)$ *of* Γ *divides the order* $|\Gamma|$, *and we have* $\exp(\Gamma) = |\Gamma|$ *if and only if every Sylow subgroup of* Γ *is cyclic.*

Lemma 2.2. *For a permutation* $\mathbb{Z}\Gamma$-*module* F, *the group* $H^2(\Gamma, F)$ *is annihilated by* $\exp(\Gamma)$.

Proof. By the definition of permutation modules, we see that a permutation $\mathbb{Z}\Gamma$-module F is isomorphic to a direct sum of modules in the form $\mathbb{Z}[\Gamma/\Gamma']$ for a subgroup Γ' of Γ. For a finite group Γ, $H^i(\Gamma, -)$ commutes with taking direct sums. Hence $H^2(\Gamma, F)$ is isomorphic to a direct sum of $\{H^2(\Gamma, \mathbb{Z}[\Gamma/\Gamma'])\}$ for subgroups $\{\Gamma'\}$ of Γ. Using Shapiro's Lemma (cf. [B, III.6.4]) and some well-known facts (cf. [B, Exercise IV.3.3]), we obtain

$$H^2(\Gamma, \mathbb{Z}[\Gamma/\Gamma']) \cong H^2(\Gamma', \mathbb{Z}) \cong H^1(\Gamma', \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Hom}(\Gamma', \mathbb{Q}/\mathbb{Z}) \cong \widehat{(\Gamma')^{\mathrm{ab}}} \cong (\Gamma')^{\mathrm{ab}}$$

where $(\Gamma')^{\mathrm{ab}}$ is the abelianization of Γ' and $\widehat{(\Gamma')^{\mathrm{ab}}}$ is its dual group. It is clear that an order of any element of $(\Gamma')^{\mathrm{ab}}$ divides $\exp(\Gamma)$. Hence $H^2(\Gamma, \mathbb{Z}[\Gamma/\Gamma'])$ is annihilated by $\exp(\Gamma)$ for any subgroup Γ' of Γ. This shows that $H^2(\Gamma, F)$ is annihilated by $\exp(\Gamma)$. $\square$

Remark 2.3. In a similar way to the proof of Lemma 2.2, we can show $H^1(\Gamma, F) = 0$ for a permutation $\mathbb{Z}\Gamma$-module F because $H^1(\Gamma, \mathbb{Z}[\Gamma/\Gamma']) \cong H^1(\Gamma', \mathbb{Z}) \cong \operatorname{Hom}(\Gamma', \mathbb{Z}) = 0$ for a subgroup Γ' of Γ. From this fact, we see that every permutation projective $\mathbb{Z}\Gamma$-module is coflasque.

Remark 2.4. If P is a permutation projective $\mathbb{Z}\Gamma$-module, then P is coflasque and satisfies that $H^2(\Gamma', P)$ is annihilated by $\exp(\Gamma')$ for all subgroups Γ' of Γ by Lemma 2.2. The author does not know whether the converse holds or not. This problem is closely related to the problem of determining whether a given module has a permutation presentation (see Corollary 3.7).

The following gives a cohomological obstruction for a $\mathbb{Z}\Gamma$-module to have permutation presentations.

Proposition 2.5. *If a* $\mathbb{Z}\Gamma$-*module* M *has a permutation presentation, then* $H^1(\Gamma, M)$ *is annihilated by* $\exp(\Gamma)$.

Proof. Let $0 \to F \to F \to M \to 0$ be a permutation presentation of M. Since $H^1(\Gamma, F) = 0$, the long exact sequence of cohomologies gives us an injection $H^1(\Gamma, M) \to H^2(\Gamma, F)$. By Lemma 2.2, $H^2(\Gamma, F)$ is annihilated by $\exp(\Gamma)$. Hence so is $H^1(\Gamma, M)$. We are done. $\quad\square$

By this proposition, we can see that if $H^1(\Gamma, M)$ is not annihilated by $\exp(\Gamma)$ then M has no permutation presentations.

Example 2.6. Let $\Gamma \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ be a group generated by two elements $\sigma, \tau \in \Gamma$ with relations $\sigma^2 = \tau^2 = 1$ and $\sigma\tau = \tau\sigma$. We have $\exp(\Gamma) = 2$. Let $M = (\mathbb{Z}/4\mathbb{Z})^3$ be a $\mathbb{Z}\Gamma$-module where an action of Γ on M is defined by

$$\sigma\big((a, b, c)\big) = (b, a, -a - b - c), \quad \tau\big((a, b, c)\big) = (c, -a - b - c, a),$$

for $a, b, c \in \mathbb{Z}/4\mathbb{Z}$. One can compute $H^1(\Gamma, M) \cong \mathbb{Z}/4\mathbb{Z}$ which is not annihilated by $\exp(\Gamma) = 2$. Hence by Proposition 2.5, we see that M has no permutation presentations.

The following example shows that the obstruction given by Proposition 2.5 is not complete.

Example 2.7. Let $\Gamma \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ be as in Example 2.6. Let $M = \mathbb{Z}/8\mathbb{Z}$ be a $\mathbb{Z}\Gamma$-module where an action of Γ on M is defined by $\sigma(m) = -m$ and $\tau(m) = 3m$ for $m \in M$. A straightforward computation shows $H^1(\Gamma, M) \cong \mathbb{Z}/2\mathbb{Z}$. Since $H^1(\Gamma, M)$ is annihilated by $\exp(\Gamma) = 2$, we cannot conclude from Proposition 2.5 that M has no permutation presentations. However, the following argument shows that M has no permutation presentations. Let $F = \mathbb{Z}[M]$ be a permutation $\mathbb{Z}\Gamma$-module, and define $\pi \colon F \to M$ by $\pi([m]) = m$. Let N_M be the kernel of the surjection π. Then we will see in Corollary 3.7 that M has a permutation presentation if and only if the $\mathbb{Z}\Gamma$-module N_M is permutation projective. We can show that $2H^2(\Gamma, N_M)$ coincides with the image of the injection $H^1(\Gamma, M) \to H^2(\Gamma, N_M)$ in the long exact sequence, and hence $2H^2(\Gamma, N_M) \neq 0$. Thus $H^2(\Gamma, N_M)$ is not annihilated by $\exp(\Gamma) = 2$, and hence N_M is not permutation projective by Lemma 2.2. Therefore M has no permutation presentations.

We use Proposition 2.5 to show the "only if" part of Theorem 1.4. The proof is inspired by [EM, Theorem 1.5].

Recall that the augmentation ideal I_Γ is the kernel of the surjection $\pi \colon \mathbb{Z}\Gamma \to \mathbb{Z}$ defined by $\pi(\gamma) = 1$ for all $\gamma \in \Gamma$. Since I_Γ is an ideal of $\mathbb{Z}\Gamma$, it is a $\mathbb{Z}\Gamma$-module. We can easily compute $H^1(\Gamma, I_\Gamma) \cong \mathbb{Z}/|\Gamma|\mathbb{Z}$.

Proposition 2.8. *If I_Γ has a permutation presentation, then every Sylow subgroup of Γ is cyclic.*

Proof. If I_Γ has a permutation presentation, then $H^1(\Gamma, I_\Gamma) \cong \mathbb{Z}/|\Gamma|\mathbb{Z}$ is annihilated by $\exp(\Gamma)$ by Proposition 2.5. This shows that $\exp(\Gamma)$ is a multiple of $|\Gamma|$. By Lemma 2.1, this implies that every Sylow subgroup of Γ is cyclic. $\quad\square$

This proposition proves the "only if" part of Theorem 1.4.

3. Coflasque modules and permutation projective modules

In this section, we see that we can reduce Theorem 1.4 to Theorem 1.6. Most of the steps of the reduction were shown for finitely generated modules in [A1] and [EM], and their proofs can be adapted to our case without big changes. We review their arguments

for readers' convenience and for the references in the rest of this paper. At the last of this section, we prove Proposition 1.3.

In [A1], Arnold introduced permutation projective dimensions (or $\mathcal{C}_1$ dimensions) of finitely generated $\mathbb{Z}\Gamma$-modules for a finite group Γ. His definition can be applied for an arbitrary $\mathbb{Z}\Gamma$-modules, and we can see that a $\mathbb{Z}\Gamma$-module M has a permutation projective dimension 0 or 1 if and only if there exists a short exact sequence

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

such that P_0 and P_1 are permutation projective $\mathbb{Z}\Gamma$-modules (see [A1] for the precise definition). Note that we can take P_0 and P_1 to be finitely generated when M is a finitely generated $\mathbb{Z}\Gamma$-module whose permutation projective dimension is 0 or 1 (see Proposition 3.6). Our first observation for the proof of Theorem 1.4 is that this condition is equivalent to have a permutation presentation.

Proposition 3.1. *A $\mathbb{Z}\Gamma$-module M has a permutation presentation if and only if there exists a short exact sequence*

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

such that P_0 and P_1 are permutation projective $\mathbb{Z}\Gamma$-modules.

Proof. This follows from the next lemma. $\qquad\qquad\square$

Lemma 3.2. *For two permutation projective $\mathbb{Z}\Gamma$-modules P_0 and P_1, there exists a permutation $\mathbb{Z}\Gamma$-module F such that $P_0 \oplus F \cong P_1 \oplus F \cong F$.*

Proof. We use the technique called "Eilenberg trick" (cf. [B, Lemma VIII.2.7]). Let F_0 and F_1 be permutation $\mathbb{Z}\Gamma$-modules containing P_0 and P_1 as direct summands, respectively. Set $F = \bigoplus_{k=1}^{\infty}(F_0 \oplus F_1)$. This $\mathbb{Z}\Gamma$-module F is a permutation module satisfying $P_0 \oplus F \cong P_1 \oplus F \cong F$. $\qquad\square$

Recall that for a $\mathbb{Z}\Gamma$-module M and a subgroup Γ' of Γ, we have

$$H^0(\Gamma', M) \cong M^{\Gamma'} := \{m \in M \mid \gamma m = m \text{ for all } \gamma \in \Gamma'\}.$$

Definition 3.3. A $\mathbb{Z}\Gamma$-module map $\pi \colon M \to N$ between $\mathbb{Z}\Gamma$-modules M, N is said to be H^0-*surjective* if its restriction $\pi \colon M^{\Gamma'} \to N^{\Gamma'}$ is surjective for every subgroup Γ' of Γ.

By setting $\Gamma' = \{1\}$, we see that an H^0-surjective map is surjective.

Lemma 3.4. *Let*

$$0 \longrightarrow M_1 \longrightarrow M_2 \overset{\pi}{\longrightarrow} M_3 \longrightarrow 0.$$

be an exact sequence of $\mathbb{Z}\Gamma$-modules. Then the following hold.

  (1) *If M_1 is coflasque, then π is H^0-surjective.*
  (2) *If π is H^0-surjective and M_2 is coflasque, then M_1 is coflasque.*
  (3) *If π is H^0-surjective and M_3 is permutation projective, then this sequence splits.*
  (4) *If M_1 is coflasque and M_3 is permutation projective, then this sequence splits.*

Proof.   (1) This follows from the long exact sequence of cohomologies.
  (2) This follows from the long exact sequence of cohomologies, and the fact that a submodule of a free $\mathbb{Z}$-module is free.
  (3) We may assume that M_3 is a permutation module, and in this case the conclusion easily follows from the definition of H^0-surjective maps.

(4) This follows from (1) and (3).

<div style="text-align:right">□</div>

Proposition 3.5. *Let*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0.$$

be an exact sequence of $\mathbb{Z}\Gamma$-modules. If both M_1 and M_3 are permutation projective, then the sequence splits, and hence $M_2 \cong M_1 \oplus M_3$ is also permutation projective.

Proof. This follows from Lemma 3.4 (4) because permutation projective modules are coflasque. □

Proposition 3.6 (cf. [A1, Theorem 3.5])**.** *A $\mathbb{Z}\Gamma$-module M has a permutation presentation if and only if for every permutation projective $\mathbb{Z}\Gamma$-module P and every H^0-surjective map $\pi\colon P \to M$, $\ker \pi$ is permutation projective.*

Proof. The proof of the "only if" part goes similarly to the proof of "Schanuel's lemma" ([B, Lemma VIII.4.2]). Suppose that a $\mathbb{Z}\Gamma$-module M has a permutation presentation

$$0 \longrightarrow F \longrightarrow F \xrightarrow{\varphi} M \longrightarrow 0$$

where F is a permutation $\mathbb{Z}\Gamma$-module. Take a permutation projective $\mathbb{Z}\Gamma$-module P and an H^0-surjective map $\pi\colon P \to M$. Let

$$N := \{(f, p) \in F \times P \mid \varphi(f) = \pi(p)\}$$

be the pull-buck of the two surjections φ and π. Then we have the following commutative diagram with exact rows and columns:

Since both F and P are permutation projective, the middle row splits and $N \cong F \oplus P$ is permutation projective by Proposition 3.5. Since π is H^0-surjective and P is permutation projective, $\ker \pi$ is coflasque by Lemma 3.4 (2). By Lemma 3.4 (4) the middle column splits and we get $N \cong \ker \pi \oplus F$. This shows that $\ker \pi$ is permutation projective.

Conversely suppose that $\ker \pi$ is permutation projective for every permutation projective $\mathbb{Z}\Gamma$-module P and every H^0-surjective map $\pi\colon P \to M$. Let us define $\pi\colon \mathbb{Z}[M] \to M$ by $\pi([m]) = m$. Since $\mathbb{Z}[M]$ is a permutation $\mathbb{Z}\Gamma$-module and π is H^0-surjective, $\ker \pi$ is permutation projective. Thus by Proposition 3.1 M has a permutation presentation. □

For a $\mathbb{Z}\Gamma$-module M, let N_M be the kernel of the H^0-surjective map $\pi\colon \mathbb{Z}[M] \to M$ defined by $\pi([m]) = m$ which is considered in the proof of Proposition 3.6. Then by Lemma 3.4 (2), N_M is coflasque. By Proposition 3.6, we get the following.

Corollary 3.7. *A $\mathbb{Z}\Gamma$-module M has a permutation presentation if and only if the coflasque $\mathbb{Z}\Gamma$-module N_M is permutation projective.*

Lemma 3.8. *For a $\mathbb{Z}\Gamma$-module N which is free as a $\mathbb{Z}$-module, there exist a permutation $\mathbb{Z}\Gamma$-module F and an injective $\mathbb{Z}\Gamma$-module map $\iota\colon N \to F$.*

Proof. Consider N as a $\mathbb{Z}$-module, and set a tensor product $F = (\mathbb{Z}\Gamma) \otimes N$ as a $\mathbb{Z}$-module. The natural action of Γ on $\mathbb{Z}\Gamma$ defines an action of Γ on F. By this action, F is a free $\mathbb{Z}\Gamma$-module. The map $\iota\colon N \to F$ defined by $\iota(n) = \sum_{\gamma \in \Gamma}[\gamma] \otimes (\gamma^{-1}n)$ is an injective $\mathbb{Z}\Gamma$-module map. $\qquad\square$

Proposition 3.9. *Every $\mathbb{Z}\Gamma$-module has a permutation presentation if and only if every coflasque $\mathbb{Z}\Gamma$-module is permutation projective.*

Proof. Suppose that every $\mathbb{Z}\Gamma$-module has a permutation presentation. Take a coflasque $\mathbb{Z}\Gamma$-module N. By Lemma 3.8, there exist a permutation $\mathbb{Z}\Gamma$-module F and an injective $\mathbb{Z}\Gamma$-module map $\iota\colon N \to F$. Set a $\mathbb{Z}\Gamma$-module M by $M = F/\iota(N)$, and denote by π the natural surjection $F \to M$. Since N is coflasque, π is H^0-surjective by Lemma 3.4 (1). Then by Proposition 3.6, $\ker \pi \cong N$ is permutation projective. This proves the "only if" part. The "if" part follows from Corollary 3.7. $\qquad\square$

Lemma 3.10 ([EM, Lemma 1.4]). *A $\mathbb{Z}\Gamma$-module M is permutation projective if and only if M is a permutation projective $\mathbb{Z}\Gamma'$-module for every Sylow subgroup Γ' of Γ.*

Proof. Clearly, if M is a permutation projective $\mathbb{Z}\Gamma$-module, then it is a permutation projective $\mathbb{Z}\Gamma'$-module for every subgroup Γ' of Γ. Let M be a $\mathbb{Z}\Gamma$-module which is permutation projective as a $\mathbb{Z}\Gamma'$-module for every Sylow subgroup Γ' of Γ. Let $\pi\colon \mathbb{Z}[M] \to M$ be an H^0-surjective map defined by $\pi([m]) = m$. By Lemma 3.4 (3), the surjection $\pi\colon \mathbb{Z}[M] \to M$ has a $\mathbb{Z}\Gamma'$-module splitting map for every Sylow subgroup Γ' of Γ. Then it has a $\mathbb{Z}\Gamma$-module splitting map (cf. [B, Proposition III.2.2 and Theorem III.10.3]). Thus M is permutation projective. $\qquad\square$

The analogous statement of the lemma above for coflasque modules also holds.

Proposition 3.11. *For a finite group Γ, every coflasque $\mathbb{Z}\Gamma$-module is permutation projective if and only if every coflasque $\mathbb{Z}\Gamma'$-module is permutation projective for every Sylow subgroup Γ' of Γ.*

Proof. Suppose that every coflasque $\mathbb{Z}\Gamma$-module is permutation projective. Take a subgroup Γ' of Γ and a coflasque $\mathbb{Z}\Gamma'$-module N. Let $M = \mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Gamma'} N$ be the induced $\mathbb{Z}\Gamma$-module from N. Using Shapiro's lemma, one can check that M is a coflasque $\mathbb{Z}\Gamma$-module. Hence by the assumption, M is a permutation projective $\mathbb{Z}\Gamma$-module. Considering as a $\mathbb{Z}\Gamma'$-module, M is also permutation projective and contains N as a direct summand. Hence N is a permutation projective $\mathbb{Z}\Gamma'$-module. This proves the "only if" part.

Conversely suppose that every coflasque $\mathbb{Z}\Gamma'$-module is permutation projective for every Sylow subgroup Γ' of Γ. Take a coflasque $\mathbb{Z}\Gamma$-module M. For every Sylow subgroup Γ' of Γ, M is a permutation projective $\mathbb{Z}\Gamma'$-module because M is coflasque as a $\mathbb{Z}\Gamma'$-module. Hence by Lemma 3.10, M is a permutation projective $\mathbb{Z}\Gamma$-module. This proves the "if" part. $\qquad\square$

We have already seen the "only if" part of Theorem 1.4 in Proposition 2.8, and the "if" part follows from Theorem 1.6 by Proposition 3.9 and Proposition 3.11. We will prove Theorem 1.6 in Section 5. We finish this section by proving Proposition 1.3.

Lemma 3.12. *A countable permutation projective module N is a direct summand of a countable permutation module.*

Proof. Let N be a countable permutation projective $\mathbb{Z}\Gamma$-module. Choose a permutation $\mathbb{Z}\Gamma$-module $\mathbb{Z}[X]$ which contains N as a direct summand, where X is a Γ-set. For each $n \in N$, let us define a finite subset $Y_n \subset X$ by $Y_n := \{x \in X \mid n_x \neq 0\}$ where $n = \sum_{x \in X} n_x[x]$. Then the set $Y = \bigcup_{n \in N} Y_n$ is a countable subset of X which is globally invariant under the action of Γ. Set $F := \mathbb{Z}[Y] \subset \mathbb{Z}[X]$. Then F is a countable permutation module containing N as a direct summand. $\qquad\square$

Proof of Proposition 1.3. Let M be a countable $\mathbb{Z}\Gamma$-module which has a permutation presentation. Then by Corollary 3.7, the coflasque $\mathbb{Z}\Gamma$-module N_M, which is defined to be the kernel of the H^0-surjective map $\pi \colon \mathbb{Z}[M] \to M$, is permutation projective. Since $\mathbb{Z}[M]$ and its submodule N_M are countable, one can take a countable permutation module F such that $\mathbb{Z}[M] \oplus F \cong N_M \oplus F \cong F$ by Lemma 3.12 and the proof of Lemma 3.2. Thus we get a countable permutation presentation

$$0 \longrightarrow F \longrightarrow F \longrightarrow M \longrightarrow 0$$

of M. $\qquad\square$

4. Projective modules over cyclic p-groups

Let p be a prime number, and q be a power of p. Let Γ be a cyclic group of order q whose generator is given by σ. We set $s \in \mathbb{Z}\Gamma$ by $s = \sum_{j=0}^{q-1} \sigma^j$. The group cohomology of a $\mathbb{Z}\Gamma$-module M has a period 2, i.e. $H^k(\Gamma, M) = H^{k+2}(\Gamma, M)$ for $k \geq 1$. Hence we are only interested in $H^1(\Gamma, M)$ and $H^2(\Gamma, M)$ which are easily computed as follows;

$$H^1(\Gamma, M) \cong \{m \in M \mid sm = 0\}/(1 - \sigma)M, \quad H^2(\Gamma, M) \cong M^\sigma/sM,$$

where $M^\sigma = \{m \in M \mid \sigma m = m\}$. The next proposition is well-known.

Proposition 4.1. *A $\mathbb{Z}\Gamma$-module M is projective if and only if M is free as a $\mathbb{Z}$-module and $H^1(\Gamma, M) = H^2(\Gamma, M) = 0$.*

For a proof of this proposition, see [Ri, Proposition 3.6] where a much more general statement was proved.

We use the following lemma in the next section.

Lemma 4.2. *Let M be a free $\mathbb{Z}\Gamma$-module. Suppose that a $\mathbb{Z}\Gamma$-submodule N of M satisfies that $p^l M \subset N$ and $H^1(\Gamma, p^{l-1}M \cap N) = 0$ for some integer l. Then there exists a free $\mathbb{Z}\Gamma$-submodule $F \subset M$ satisfying that*

- *M/F is free as a $\mathbb{Z}$-module,*
- *$p^{l-1}M \subset F + N$, and*
- *$F \cap N = p^l F$.*

Proof. We set $M' = (p^{l-1}M)/(p^{l-1}M \cap N)$ and let $\pi \colon p^{l-1}M \to M'$ be the natural surjection:

$$0 \longrightarrow p^{l-1}M \cap N \longrightarrow p^{l-1}M \xrightarrow{\ \pi\ } M' \longrightarrow 0.$$

Since $p^l M \subset N$, M' is a $(\mathbb{Z}/p\mathbb{Z})\Gamma$-module. Since $H^2(\Gamma, p^{l-1}M) = 0$ and $H^3(\Gamma, p^{l-1}M \cap N) = H^1(\Gamma, p^{l-1}M \cap N) = 0$, we have $H^2(\Gamma, M') = 0$. Let X be a basis of M as a $\mathbb{Z}\Gamma$-module. Then $s\pi(p^{l-1}X)$ generates sM' which coincides with $(M')^\sigma$ because $H^2(\Gamma, M') = 0$. Choose a subset $Y \subset X$ such that $s\pi(p^{l-1}Y)$ is a basis of $(M')^\sigma$ as a $\mathbb{Z}/p\mathbb{Z}$-module. Let $F := (\mathbb{Z}\Gamma)Y$ be the free $\mathbb{Z}\Gamma$-submodule of M generated by Y. Then M/F is free as a $\mathbb{Z}$-module (actually as a $\mathbb{Z}\Gamma$-module). We set $Y_0 = \bigcup_{k=0}^{q-1}(\sigma - 1)^k Y$ which is a basis of F as a $\mathbb{Z}$-module. By noticing $s = (\sigma - 1)^{q-1}$ in $(\mathbb{Z}/p\mathbb{Z})\Gamma$, we see that $\pi(p^{l-1}Y_0) = \bigcup_{k=0}^{q-1}(\sigma - 1)^k \pi(p^{l-1}Y)$ is a basis of M' as a $\mathbb{Z}/p\mathbb{Z}$-module. This shows that $p^{l-1}M \subset p^{l-1}F + N \subset F + N$. This also shows that if $x \in p^{l-1}F$ satisfies $\pi(x) = 0$ then $x \in p^l F$. The proof ends once we show $F \cap N = p^l F$. Clearly we have $F \cap N \supset p^l F$. To derive a contradiction, assume that there exists $x_0 \in (F \cap N) \setminus p^l F$. Choose a natural number k such that $p^k x_0 \in p^{l-1}F \setminus p^l F$. Since $p^k x_0 \in N$, we have $\pi(p^k x_0) = 0$. This contradicts the fact above. Thus we have $F \cap N = p^l F$. We are done. $\qquad\square$

5. Coflasque modules over cyclic p-groups

In this section, we show Theorem 1.6, and complete the proof of Theorem 1.4.

Let p be a prime number, and K be a positive integer. Let Γ be the cyclic group of order p^K whose generator is given by $\sigma \in \Gamma$. Let us take $k \in \{0, 1, \ldots, K\}$. We set $\sigma_k = \sigma^{p^k}$. Thus $\sigma_0 = \sigma$, $\sigma_K = 1$ and $\sigma_{k+1} = (\sigma_k)^p$. Let $\Gamma_k \subset \Gamma$ be the subgroup generated by σ_k. We have $\Gamma_k \cong \mathbb{Z}/p^{K-k}\mathbb{Z}$ and $\Gamma/\Gamma_k \cong \mathbb{Z}/p^k\mathbb{Z}$. For $l \in \{k, \ldots, K\}$, we set an element $s_k^{(l)} \in \mathbb{Z}\Gamma$ by $s_k^{(l)} = \sum_{j=0}^{p^{l-k}-1} \sigma_k^j$. We have

$$(1 - \sigma_k)s_k^{(l)} = 1 - \sigma_l, \quad s_k^{(l)} s_l^{(m)} = s_k^{(m)}$$

for $k \leq l \leq m$. When $l = K$, the element $s_k^{(K)} \in \mathbb{Z}\Gamma$ is simply denoted by s_k.

Every $\mathbb{Z}\Gamma$-module M can also be considered as a $\mathbb{Z}\Gamma_k$-module for $k = 0, 1, \ldots, K$, and we have

$$H^1(\Gamma_k, M) \cong \{m \in M \mid s_k m = 0\}/(1 - \sigma_k)M, \quad H^2(\Gamma_k, M) \cong M^{\sigma_k}/s_k M.$$

By a $\mathbb{Z}(\Gamma/\Gamma_k)$-module, we mean a $\mathbb{Z}\Gamma$-module M with $M^{\sigma_k} = M$. Since $\{\Gamma_k\}_{k=0}^K$ exhausts all subgroups of Γ, a permutation $\mathbb{Z}\Gamma$-module is nothing but a direct sum of free $\mathbb{Z}(\Gamma/\Gamma_k)$-modules for $k = 0, 1, \ldots, K$.

Definition 5.1. Let $k = 0, 1, \ldots, K$. We write $M \in \mathcal{C}_k$ if M is a coflasque $\mathbb{Z}\Gamma$-module such that M^{σ_k} is a projective $\mathbb{Z}(\Gamma/\Gamma_k)$-module.

Note that $M \in \mathcal{C}_0$ if and only if M is a coflasque $\mathbb{Z}\Gamma$-module, and $M \in \mathcal{C}_K$ if and only if M is a projective $\mathbb{Z}\Gamma$-module.

Lemma 5.2. *Let $k \in \{1, 2, \ldots, K\}$. For a coflasque $\mathbb{Z}\Gamma$-module M, the following conditions are equivalent;*

(i) $M \in \mathcal{C}_k$,
(ii) $p^{K-k}M^{\sigma_{k-1}} \subset s_{k-1}M$,
(iii) $M^{\sigma_{k-1}} = s_{k-1}^{(k)}M^{\sigma_k}$,
(iv) $M^{\sigma_0} = s_0^{(k)}M^{\sigma_k}$.

Proof. (i)$\Rightarrow$(ii): If M^{σ_k} is a projective $\mathbb{Z}(\Gamma/\Gamma_k)$-module, then we have $M^{\sigma_{k-1}} = s_{k-1}^{(k)}M^{\sigma_k}$. Hence we get

$$p^{K-k}M^{\sigma_{k-1}} = p^{K-k}s_{k-1}^{(k)}M^{\sigma_k} = s_{k-1}M^{\sigma_k} \subset s_{k-1}M.$$

(ii)$\Rightarrow$(iii): Take $m \in M^{\sigma_{k-1}}$. By (ii), there exists $x \in M$ with $p^{K-k}m = s_{k-1}x$. Then we get $s_k(m - s_{k-1}^{(k)}x) = 0$. Since $H^1(\Gamma_k, M) = 0$, we can find $y \in M$ such that $m - s_{k-1}^{(k)}x = (1 - \sigma_k)y = s_{k-1}^{(k)}(1 - \sigma_{k-1})y$. Set $m' = x + (1 - \sigma_{k-1})y \in M$. Then we have $m = s_{k-1}^{(k)}m'$. Since $m \in M^{\sigma_{k-1}}$, we see $m' \in M^{\sigma_k}$. Thus we have shown $M^{\sigma_{k-1}} \subset s_{k-1}^{(k)}M^{\sigma_k}$. The converse inclusion is obvious.

(iii)$\Rightarrow$(iv): First we show that $M^{\sigma_l} = s_l^{(l+1)}M^{\sigma_{l+1}}$ holds for $l = 0, 1, \ldots, k - 1$. By (iii), this holds for $l = k - 1$. Suppose that this holds for l, and we show the equality for $l - 1$. Take $m \in M^{\sigma_{l-1}}$. By the assumption, there exists $x \in M^{\sigma_{l+1}}$ such that $m = s_l^{(l+1)}x$. Then

$$s_l^{(l+1)}(m - s_{l-1}^{(l)}x) = pm - s_{l-1}^{(l)}m = 0$$

because $m \in M^{\sigma_{l-1}}$. Since $H^1(\Gamma_l, M) = 0$, we can find $y \in M$ with $m - s_{l-1}^{(l)}x = (1-\sigma_l)y = s_{l-1}^{(l)}(1-\sigma_{l-1})y$. Set $m' = x + (1-\sigma_{l-1})y$. Then we have $m = s_{l-1}^{(l)}m'$. The fact $m \in M^{\sigma_{l-1}}$ implies $m' \in M^{\sigma_l}$. Thus we get $M^{\sigma_{l-1}} \subset s_{l-1}^{(l)}M^{\sigma_l}$, and the converse inclusion is obvious. We showed that $M^{\sigma_l} = s_l^{(l+1)}M^{\sigma_{l+1}}$ holds for $l = 0, 1, \ldots, k - 1$. Hence

$$M^{\sigma_0} = s_0^{(1)}M^{\sigma_1} = s_0^{(1)}s_1^{(2)}M^{\sigma_2} = \cdots\cdots = s_0^{(1)}s_1^{(2)}\cdots s_{k-1}^{(k)}M^{\sigma_k} = s_0^{(k)}M^{\sigma_k}.$$

(iv)$\Rightarrow$(i): The condition (iv) is nothing but $H^2(\Gamma/\Gamma_k, M^{\sigma_k}) = 0$. On the other hand, we have $H^1(\Gamma/\Gamma_k, M^{\sigma_k}) = 0$ for any coflasque $\mathbb{Z}\Gamma$-module M because there exists an injection $H^1(\Gamma/\Gamma_k, M^{\sigma_k}) \to H^1(\Gamma, M)$ induced by the inflation (see [Se, Proposition VII.4]). Thus M^{σ_k} is a projective $\mathbb{Z}(\Gamma/\Gamma_k)$-module by Proposition 4.1. $\square$

Lemma 5.3. Let $k \in \{0, 1, \ldots, K - 1\}$. Let $\tilde{M} \in \mathcal{C}_k$, and set $M = \tilde{M}^{\sigma_k}$ and $N = s_k\tilde{M}$ which are $\mathbb{Z}(\Gamma/\Gamma_k)$-modules. Then $p^{K-k}M \subset N \subset M$ and $H^1(\Gamma/\Gamma_k, p^{K-k-1}M \cap N) = 0$.

Proof. Clearly we have $p^{K-k}M \subset N \subset M$. Take $x \in p^{K-k-1}M \cap N$ with $s_0^{(k)}x = 0$. Since M is a projective $\mathbb{Z}(\Gamma/\Gamma_k)$-module, we have $H^1(\Gamma_0/\Gamma_k, p^{K-k-1}M) = 0$. Hence there exists $y \in M$ with $x = (1 - \sigma_0)p^{K-k-1}y$. Since $x \in N = s_k\tilde{M}$, there exists $x = s_km$ with $m \in \tilde{M}$. We have

$$s_0 m = s_0^{(k)}x = s_0^{(k)}(1 - \sigma_0)p^{K-k-1}y = (1 - \sigma_k)p^{K-k-1}y = 0$$

because $y \in M = \tilde{M}^{\sigma_k}$. Since $H^1(\Gamma_0, \tilde{M}) = 0$, there exists $z \in \tilde{M}$ with $m = (1 - \sigma_0)z$. Then

$$s_{k+1}(1 - \sigma_0)(y - s_k^{(k+1)}z) = (1 - \sigma_0)p^{K-k-1}y - s_k(1 - \sigma_0)z = x - x = 0.$$

Since $H^1(\Gamma_{k+1}, \tilde{M}) = 0$, there exists $m_0 \in \tilde{M}$ such that

$$(1 - \sigma_0)(y - s_k^{(k+1)}z) = (1 - \sigma_{k+1})m_0 = (1 - \sigma_0)s_k^{(k+1)}s_0^{(k)}m_0.$$

Set $z' = z + s_0^{(k)}m_0 \in \tilde{M}$. Then we have $(1 - \sigma_0)y = (1 - \sigma_0)s_k^{(k+1)}z'$. Since $y \in M = \tilde{M}^{\sigma_k}$, $s_k^{(k+1)}z' \in \tilde{M}^{\sigma_k}$. Hence the element

$$x' := p^{K-k-1}s_k^{(k+1)}z' = s_{k+1}(s_k^{(k+1)}z') = s_kz'$$

is in $p^{K-k-1}M \cap N$. We have

$$(1 - \sigma_0)x' = (1 - \sigma_0)p^{K-k-1}(s_k^{(k+1)}z') = (1 - \sigma_0)p^{K-k-1}y = x$$

This shows $H^1(\Gamma/\Gamma_k, p^{K-k-1}M \cap N) = 0$. We are done. $\square$

Proposition 5.4. *Let $k \in \{0, 1, \ldots, K-1\}$. For $M \in \mathcal{C}_k$, there exists a short exact sequence of $\mathbb{Z}\Gamma$-modules*

$$0 \longrightarrow F \longrightarrow M \oplus P \longrightarrow M' \longrightarrow 0$$

where F is a free $\mathbb{Z}(\Gamma/\Gamma_k)$-module, P is a projective $\mathbb{Z}(\Gamma/\Gamma_k)$-module, and $M' \in \mathcal{C}_{k+1}$.

Proof. Since $M \in \mathcal{C}_k$, M^{σ_k} is a projective $\mathbb{Z}(\Gamma/\Gamma_k)$-module. Hence there exists a projective $\mathbb{Z}(\Gamma/\Gamma_k)$-module P such that $M^{\sigma_k} \oplus P$ is a free $\mathbb{Z}(\Gamma/\Gamma_k)$-module. Set $\tilde{M} := M \oplus P$. Then $\tilde{M}$ is a coflasque $\mathbb{Z}\Gamma$-module, and $\tilde{M}^{\sigma_k} = M^{\sigma_k} \oplus P$ is a free $\mathbb{Z}(\Gamma/\Gamma_k)$-module. By Lemma 5.3, we can apply Lemma 4.2 to get a free $\mathbb{Z}(\Gamma/\Gamma_k)$-module $F \subset \tilde{M}^{\sigma_k}$ such that $\tilde{M}^{\sigma_k}/F$ is free as a $\mathbb{Z}$-module, $p^{K-k-1}\tilde{M}^{\sigma_k} \subset F + s_k\tilde{M}$ and $F \cap s_k\tilde{M} = p^{K-k}F$. Set $M' = \tilde{M}/F$. The proof ends if we show $M' \in \mathcal{C}_{k+1}$. Since $\tilde{M}^{\sigma_k}/F$ and $\tilde{M}/\tilde{M}^{\sigma_k} \cong (1 - \sigma_k)\tilde{M} \subset \tilde{M}$ are free as $\mathbb{Z}$-modules, $M' = \tilde{M}/F$ is also free as a $\mathbb{Z}$-module. To show that $H^1(\Gamma_l, M') = 0$ for $l = 0, 1, \ldots, K$, it suffices to see by the long exact sequence of cohomologies that the map $H^2(\Gamma_l, F) \to H^2(\Gamma_l, \tilde{M})$ is injective. Let us take $f \in F^{\sigma_l}$ such that $f = s_l m$ for some $m \in \tilde{M}$. We will show $f \in s_l F$. First consider the case $l \leq k$. Then $f = s_l m = s_k(s_l^{(k)}m) \in s_k\tilde{M}$. From $F \cap s_k\tilde{M} = p^{K-k}F$ we can find $f' \in F$ with $f = p^{K-k}f'$. Since $f \in F^{\sigma_l}$, f' is also in F^{σ_l}. Since F is a free $\mathbb{Z}(\Gamma/\Gamma_k)$-module, we can find $g \in F$ such that $f' = s_l^{(k)}g$. Then $s_l g = s_k f' = p^{K-k}f' = f$. Thus we have shown $f \in s_l F$. Next we consider the case $l > k$. We have $p^{l-k}f = s_k^{(l)}f \in s_k\tilde{M}$. Hence there exists $g \in F$ with $p^{l-k}f = p^{K-k}g$. Then we get $f = p^{K-l}g = s_l g \in s_l F$. Thus we have shown that the map $H^2(\Gamma_l, F) \to H^2(\Gamma_l, \tilde{M})$ is injective. Therefore M' is a coflasque $\mathbb{Z}\Gamma$-module. Since F is a free $\mathbb{Z}(\Gamma/\Gamma_k)$-module, we have $H^1(\Gamma_k, F) = 0$. This implies that $\tilde{M}^{\sigma_k} \to (M')^{\sigma_k}$ is surjective. From this fact and $p^{K-k-1}\tilde{M}^{\sigma_k} \subset F + s_k\tilde{M}$, we get $p^{K-k-1}(M')^{\sigma_k} \subset s_k M'$. Therefore Lemma 5.2 shows $M' \in \mathcal{C}_{k+1}$. This completes the proof. $\square$

Proof of Theorem 1.6. Take a coflasque $\mathbb{Z}\Gamma$-module M_0. We have $M_0 \in \mathcal{C}_0$. By applying Proposition 5.4 inductively, we get $\mathbb{Z}\Gamma$-modules $\{F_k, P_k\}_{k=0}^{K-1}$ and $\{M_k\}_{k=1}^{K}$ where F_k is a free $\mathbb{Z}(\Gamma/\Gamma_k)$-module, P_k is a projective $\mathbb{Z}(\Gamma/\Gamma_k)$-module and $M_k \in \mathcal{C}_k$ such that there exist short exact sequences

$$(*) \qquad 0 \longrightarrow F_k \longrightarrow M_k \oplus P_k \longrightarrow M_{k+1} \longrightarrow 0.$$

Since $M_K \in \mathcal{C}_K$ is a projective $\mathbb{Z}\Gamma$-module, there exists a projective $\mathbb{Z}\Gamma$-module P_K such that $F_K = M_K \oplus P_K$ is a free $\mathbb{Z}\Gamma$-module. We will show that

$$M_k \oplus \bigoplus_{l=k}^{K} P_l \cong \bigoplus_{l=k}^{K} F_l$$

holds for $k = 0, 1, \ldots, K$ by the induction on k from above. When $k = K$, it is OK. Suppose that we get the isomorphism for $k + 1$. By adding the module $\bigoplus_{l=k+1}^{K} P_l$ to the sequence $(*)$, we get the short exact sequence

$$0 \longrightarrow F_k \longrightarrow M_k \oplus \bigoplus_{l=k}^{K} P_l \longrightarrow M_{k+1} \oplus \bigoplus_{l=k+1}^{K} P_l \longrightarrow 0.$$

Since both F_k and $M_{k+1} \oplus \bigoplus_{l=k+1}^{K} P_l \cong \bigoplus_{l=k+1}^{K} F_l$ are permutation modules, this sequence splits and we get the desired isomorphism $M_k \oplus \bigoplus_{l=k}^{K} P_l \cong \bigoplus_{l=k}^{K} F_l$ by Proposition 3.5. Hence we obtain $M_0 \oplus \bigoplus_{l=0}^{K} P_l \cong \bigoplus_{l=0}^{K} F_l$. Therefore M_0 is permutation projective. This completes the proof. $\square$

Combining Propositions 2.8, 3.9, 3.11 and Theorem 1.6, we get Theorem 1.4.

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